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ON UNSYMMETRICAL ADJUSTMENTS, AND THEIR LIMITS.

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DENOTING any series of equidistant terms, unlimited in length, by

$$\dots . u_{-2}, u_{-1}, u_0, u_1, u_2 \dots . \text{\&c.},$$

let it be adjusted, for example, by the formula

$$\begin{aligned} u'_0 &= l_0u_0+l_1\, u_1\, +l_2\, u_2\, +l_3\, u_3\, +l_4\, u_4 \\ &\quad +l_{-1}u_{-1}+l_{-2}u_{-2}+l_{-3}u_{-3}+l_{-4}u_{-4}, \end{aligned} \tag{1}$$

which differs from the formulas heretofore used, in that the coefficients l equidistant from the middle, as l_3 and l_{-3} for instance, are not necessarily equal to each other, so that when they are all regarded as ordinates to a curve, the curve need not be symmetrical on each side of l_0. If such a formula is to include more terms on one side of u_0 than on the other, we have simply to suppose that the coefficients of the absent terms are zero. Having obtained a number of adjusted terms

$$\dots . u'_{-2},\quad u'_{-1},\quad u'_0,\quad u'_1,\quad u'_2, \dots .$$

let them be adjusted again by another unsymmetrical formula,

$$u''_0 = L_0u'_0+L_1u'_1+L_2u'_2+L_{-1}u'_{-1}+L_{-2}u'_{-2}. \tag{2}$$

By a process of substitution quite similar to that followed in my article in the ANALYST for May 1878, p. 65, where the formulas were symmetrical, we here get the following expression;

$$\left.\begin{array}{l} u''_0 = (L_0l_0+L_1l_{-1}+L_2l_{-2}+L_{-1}l_1+L_{-2}l_2)u_0 \\ \quad + (L_0l_1+L_1l_0\, +L_2l_{-1}+L_{-1}l_2+L_{-2}l_3)u_1 \\ \quad + (L_0l_2+L_1l_1\, +L_2l_0\, +L_{-1}l_3+L_{-2}l_4)u_2 \\ \quad + (L_0l_3+L_1l_2\, +L_2l_1\, +L_{-1}l_4)u_3 \\ \quad + (L_0l_4+L_1l_3\, +L_2l_2)u_4 \\ \qquad\qquad +L_1l_4\, +L_2l_3)u_5+L_2l_4u_6 \\ \quad + \text{terms containing } u_{-1},\ u_{-2}, \dots . u_{-6}. \end{array}\right\} \tag{3}$$

The coefficients of u_{-1}, u_{-2} &c., will be the same as those of u_1, u_2 &c. respectively, except that all the sub-indices of L and l will have their signs changed. This formula becomes identical with (3) of the former article when the negative sub-indices of L and l are changed to positive. It is a resultant formula of adjustment, equivalent to (1) and (2) used in succession.

Let the coefficients of the several terms u in (1) and (2) become those of the powers of z in the two polynomials

$$l_{-4}\, +l_{-3}z\, +l_{-2}z^2+l_{-1}z^3+l_0z^4+l_1z^5+l_2z^6+l_3z^7+l_4z^8,$$

$$L_{-2}+L_{-1}z+L_0z^2+L_1z^3+L_2z^4.$$

When these two are multiplied together, it will be found that the coefficients of the several powers of z in the product are the same as those of the several terms u in the formula (3). Like results are obtained when formulas other than (1) and (2), that is, containing different numbers of terms, are successively used, and we generalize the fact thus. Let any two component formulas be

$$\left.\begin{aligned} u'_0 &= l_0 u_0 + l_1 u_1 + l_2 u_2 + \dots\dots + l_m u_m \\ &\quad + l_{-1} u_{-1} + l_{-2} u_{-2} + \dots\dots + l_{-m} u_{-m} \\ u'_0 &= L_0 u_0 + L_1 u_1 + L_2 u_2 + \dots\dots + L_n u_n \\ &\quad + L_{-1} u_{-1} + L_{-2} u_{-2} + \dots\dots + L_{-n} u_{-n} \end{aligned}\right\} \qquad (4)$$

When the given series is adjusted by these two in succession, no matter which is used first, the result will be the same as that of a single adjustment made by a formula whose coefficients are those of the powers of z in the product of the two polynomials

$$\left.\begin{aligned} &l_{-m} + l_{-m+1} z + l_{-m+2} z^2 + \dots + l_0 z^m + l_1 z^{m+1} + \dots + l_m z^{2m}, \\ &L_{-n} + L_{-n+1} z + L_{-n+2} z^2 + \dots + L_0 z^n + L_1 z^{n+1} + \dots + L_n z^{2n}. \end{aligned}\right\} \quad (5)$$

In the resultant formula, the coefficient of the term u_0 is that one which includes the product $L_0 l_0$. It may be noticed that the coefficients in the product of the polynomial factors are independent of the particular exponents assigned to z in the two factors, provided that each series of exponents is an arithmetical progression, and that both progressions increase by the same constant difference. We might therefore use instead

$$\left.\begin{aligned} &l_{-m} z^{-m} + \dots\dots + l_{-1} z^{-1} + l_0 z^0 + l_1 z + \dots\dots + l_m z^m, \\ &L_{-n} z^{-n} + \dots\dots + L_{-1} z^{-1} + L_0 z^0 + L_1 z + \dots\dots + L_n z^n, \end{aligned}\right\} \qquad (6)$$

where the indices of z are the same as the sub-indices of l, L and u.

It follows from the foregoing that if a series is adjusted k times in succession by the single formula

$$\begin{aligned} u'_0 &= \lambda_0 u_0 + \lambda_1 u_1 + \lambda_2 u_2 + \dots\dots + \lambda_m u_m \\ &\quad + \lambda_{-1} u_{-1} + \lambda_{-2} u_{-2} + \dots\dots + \lambda_{-m} u_{-m}, \end{aligned} \qquad (7)$$

and if the equivalent or resultant formula is denoted by

$$\begin{aligned} u_0^{(k)} &= l_0 u_0 + l_1 u_1 + l_2 u_2 + \dots\dots + l_{km} u_{km} \\ &\quad + l_{-1} u_{-1} + l_{-2} u_{-2} + \dots\dots + l_{-km} u_{-km}, \end{aligned} \qquad (8)$$

then the coefficients l will be the same as the coefficients of the powers of z in the expansion of

$$(\lambda_{-m} + \lambda_{-m+1} z + \lambda_{-m+2} z^2 + \dots\dots + \lambda_0 z^m + \lambda_1 z^{m+1} + \dots\dots + \lambda_m z^{2m})^k. \quad (9)$$

A convenient general form of this expansion, known as the Multinomial Formula, may be found in Davies and Peck's *Dictionary of Mathematics.* (For a demonstration, but in less convenient form, see Price's *Calculus,* Vol. I, p. 161.) It is such that if we write

$$(a_0+a_1z+a_2z^2+\ldots\ldots+a_nz^n)^k = B_0+B_1z+B_2z^2+\ldots\ldots+B_{kn}z^{kn}, \quad (10)$$

then any group of $n+1$ coefficients B in the expansion, taken from B_{r-n} up to B_r inclusive, are evidently connected by the relation

$$B_r = \frac{1}{a_0 r}\left\{ (k-r+1)a_1B_{r-1}+(2k-r+2)a_2B_{r-2}+\ldots+(nk-r+n)a_nB_{r-n} \right\} \quad (11.)$$

This relation, by the way, holds also for the expansion extended backward to terms in z^{-1}, z^{-2} &c., provided that their coefficients B_{-1}, B_{-2} &c., are made zero, so that, remembering that $B_0 = a_0^k$, we can find the other coefficients B_1, B_2 &c., successively from (11), by giving to r the values 1, 2, 3 &c., in succession. Formulas (10) and (11) may therefore be regarded as expressing the multinomial theorem in a concise form.

Now let us take $n = 2m$, and change the notation for the coefficients to correspond with (8) and (9).

For $a_0, a_1, a_2 \ldots\ldots a_n$ write $\lambda_{-m}, \lambda_{-m+1} \ldots\ldots \lambda_{-1}, \lambda_0, \lambda_1 \ldots\ldots \lambda_m$.

For $B_0, B_1, B_2 \ldots B_{kn}$ write $l_{-km}, l_{-km+1} \ldots\ldots l_{-1}, l_0, l_1 \ldots\ldots l_{km}$.

Also let the place of the last one of any group of $2m+1$ coefficients in the expansion be denoted, not by its rank r reckoned from the first term, but by its rank $i+m$ reckoned from the middle term l_0; so that l_i is the middle term of the group, and i is its rank reckoned from l_0. This gives

$$r = km + i + m,$$

and the group

$$B_r, B_{r-1} \ldots B_{r-n}$$

is now

$$l_{i+m}, l_{i+m-1} \ldots\ldots l_{i+1}, l_i, l_{i-1}, \ldots\ldots l_{i-m}.$$

Substituting the above in (11), we get

$$\begin{aligned} &-(mk+i+m)\lambda_{-m}l_{i+m}-[(m-1)k+i+m-1]\lambda_{-m+1}l_{i+m-1}-\ldots\ldots \\ &-(\,k+i+1\,)\lambda_{-1}l_{i+1}-i\lambda_0l_i+(k-i+1)\lambda_1l_{i-1}+(2k-i+2)\lambda_2l_{i-2}+\ldots \\ &\qquad +(mk-i+m)\lambda_ml_{i-m} = 0, \end{aligned}$$

and consequently

$$\frac{(\lambda_1l_{i-1}-\lambda_{-1}l_{i+1})+2(\lambda_2l_{i-2}-\lambda_{-2}l_{i+2})+\ldots+m(\lambda_ml_{i-m}-\lambda_{-m}l_{i+m})}{\lambda_0l_i+(\lambda_1l_{i-1}+\lambda_{-1}l_{i+1})+(\lambda_2l_{i-2}+\lambda_{-2}l_{i+2})+\ldots+(\lambda_ml_{i-m}+\lambda_{-m}l_{i+m})} = \frac{i}{k+1} \quad (12.)$$

This, like (11), is simply a form of the multinomial theorem. Our changes of notation have not affected its generality, for if the number of terms in the first member of (10) were an even number $2m$, we could always make it the odd number $2m+1$ without altering its value, by adding a term with coefficient zero. As has been seen, (12) expresses not only the relation between any $2m+1$ consecutive coefficients l of the powers of z in the expansion of (9), but also the relation between any $2m+1$ consecutive coefficients

l, in the resultant adjustment formula (8), which is the equivalent of k successive applications of (7). When we suppose (7) to be symmetrical, so that $\lambda_{-1} = \lambda_1$, $\lambda_{-2} = \lambda_2$ &c., then (12) becomes identical with (3) of my article in ANALYST for Sept., 1878, and a demonstration is found for the property which was there inferred only from a number of isolated cases.

We will now proceed to determine the limiting curve followed by the coefficients l when the number of applications of the unsymmetrical formula (7) becomes infinite. This is a general problem, including the special case of a symmetrical formula. The process will be similar to that employed for the special case, in my article just mentioned.

Let the finite differences of the series l_i, l_{i+1} &c., be

$$\begin{array}{ll} \Delta_1 = l_{i+1} - l_i, & \Delta_3 = l_{i+3} - 3l_{i+2} + 3l_{i+1} - l_i, \\ \Delta_2 = l_{i+2} - 2l_{i+1} + l_i, & \Delta_4 = l_{i+4} - 4l_{i+3} + 6l_{i+2} - 4l_{i+1} + l_i, \end{array}$$
$$\Delta_5 = l_{i+5} - 5l_{i+4} + 10l_{i+3} - 10l_{i+2} + 5l_{i+1} - l_i,$$

and let differences higher than Δ_5 be neglected, so that Δ_5 is regarded as constant for any six consecutive coefficients l. Thus we find

$$\begin{array}{ll} l_{i+1} = l_i + \Delta_1, & l_{i-1} = l_i - \Delta_1 + \Delta_2 - \Delta_3 + \Delta_4 - \Delta_5, \\ l_{i+2} = l_i + 2\Delta_1 + \Delta_2, & l_{i-2} = l_i - 2\Delta_1 + 3\Delta_2 - 4\Delta_3 + 5\Delta_4 - 6\Delta_5, \\ l_{i+3} = l_i + 3\Delta_1 + 3\Delta_2 + \Delta_3, & l_{i-3} = l_i - 3\Delta_1 + 6\Delta_2 - 10\Delta_3 + 15\Delta_4 - 21\Delta_5, \\ \text{\&c.,} & \text{\&c.} \end{array}$$

The law of these expressions is such that we have in general

$$\left.\begin{array}{l} l_{i+n} = l_i + n\Delta_1 + \frac{1}{2}n(n-1)\Delta_2 + \frac{1}{2\,.\,3}\,n(n-1)(n-2)\Delta_3 \\ \qquad\qquad + \frac{1}{2\,.\,3\,.\,4}\,n(n-1)(n-2)(n-3)\Delta_4 + \text{\&c.,} \\ l_{i-n} = l_i - n\Delta_1 + \frac{1}{2}n(n+1)\Delta_2 - \frac{1}{2\,.\,3}\,n(n+1)(n+2)\Delta_3 \\ \qquad\qquad + \frac{1}{2\,.\,3\,.\,4}\,n(n+1)(n+2)(n+3)\Delta_4 - \text{\&c.,} \end{array}\right\} (13)$$

and they will be the same if differences higher than Δ_5 are taken into account. Substitute them in (12), and write for brevity

$$\left.\begin{array}{l} b_0 = \lambda_0 + (\lambda_1 + \lambda_{-1}) + \ (\lambda_2 + \lambda_{-2}) + \ldots\ldots + \ \ (\lambda_m + \lambda_{-m}) \\ b_1 = \quad 1\ (\lambda_1 - \lambda_{-1}) + 2\,(\lambda_2 - \lambda_{-2}) + \ldots\ldots + m\,(\lambda_m - \lambda_{-m}) \\ b_2 = \quad 1^2\,(\lambda_1 + \lambda_{-1}) + 2^2(\lambda_2 + \lambda_{-2}) + \ldots\ldots + m^2(\lambda_m + \lambda_{-m}) \\ b_3 = \quad 1^3\,(\lambda_1 - \lambda_{-1}) + 2^3(\lambda_2 - \lambda_{-2}) + \ldots\ldots + m^3(\lambda_m - \lambda_{-m}) \\ b_4 = \quad 1^4\,(\lambda_1 + \lambda_{-1}) + 2^4(\lambda_2 + \lambda_{-2}) + \ldots\ldots + m^4(\lambda_m + \lambda_{-m}) \\ \qquad\qquad \text{\&c.,} \qquad\qquad\qquad\qquad\qquad \text{\&c.} \end{array}\right\} \quad (14)$$

It will be seen that b_2, b_4 &c., here are not the same as they were at p. 130 of my former article, since they include the λ with negative sub-indices.

Denoting the numerator and denominator of the first member of (12) by N and D, we get

$$\left.\begin{aligned}
N &= b_1 l_i - b_2\Delta_1 + \tfrac{1}{2}(b_2+b_3)\Delta_2 - \tfrac{1}{2\,.\,3}(2b_2+3b_3+b_4)\Delta_3\\
&\qquad + \tfrac{1}{2\cdot 3\cdot 4}(6b_2+11b_3+6b_4+b_5)\Delta_4 - \&c.,\\
D &= b_0 l_i - b_1\Delta_1 + \tfrac{1}{2}(b_1+b_2)\Delta_2 - \tfrac{1}{2\,.\,3}(2b_1+3b_2+b_3)\Delta_3\\
&\qquad + \tfrac{1}{2\cdot 3\cdot 4}(6b_1+11b_2+6b_3+b_4)\Delta_4 - \&c.,\\
\frac{N}{D} &= \frac{i}{k+1}.
\end{aligned}\right\}\quad(15)$$

It will be found that in the coefficient of any difference Δ_m the numerical coefficients of the b within the parentheses are the same as those of the powers of n in the product of the factors

$$(1+n),\ (2+n),\ \ldots .\ [(m-1)+n],$$

so that the above expressions for N and D might be readily extended as far as desired. Now when k becomes infinite, and the successive values of l are regarded as consecutive ordinates to a limiting curve, we have

$$l_i = y,\qquad \Delta_1 = dy,\qquad \Delta_2 = d^2y,\qquad \Delta_3 = d^3y,\ \&c.$$

Let l_0 be taken as the axis of Y. The abscissa x corresponding to any y, which before was $x = i\Delta x$, becomes $x = idx$ at the limit, when the ordinates are set close together so as to be consecutive. Thus (15) gives

$$\frac{b_1y-b_2dy+\frac{1}{2}(b_2+b_3)d^2y-\frac{1}{2\,.\,3}(2b_2+3b_3+b_4)d^3y+\&c.}{b_0y-b_1dy+\frac{1}{2}(b_1+b_2)d^2y-\frac{1}{2\,.\,3}(2b_1+3b_2+b_3)d^3y+\&c.} = \frac{x}{(k+1)dx},\quad(16)$$

as the differential equation of the limiting curve. The order of this equation and the nature of the curve will depend upon the values of the constants b.

To show their significance, suppose that the series of terms u_0, u_1 &c., to which the adjusting process is applied, is of algebraic form and therefore represented by equidistant ordinates u to the curve

$$u = A_0+A_1x+A_2x^2+A_3x^3+\&c.,\quad(17)$$

whose degree is equal to the order of the series. The origin may be taken at any term we please, and we take it at u_0, so that $u_0 = A_0$. The constant interval between the ordinates may be the unit of x. Thus we have

$$\begin{array}{ll}
u_1 = A_0+\ A_1+\ A_2+\ \&c., & u_{-1} = A_0-\ A_1+\ A_2-\ \&c.,\\
u_2 = A_0+2A_1+2^2A_2+2^3A_3+.., & u_{-2} = A_0-2A_1+2^2A_2-2^3A_3+\ldots,\\
u_3 = A_0+3A_1+3^2A_2+3^3A_3+.., & u_{-3} = A_0-3A_1+3^2A_2-3^3A_3+\ldots,\\
\qquad\&c., & \qquad\&c.,
\end{array}$$

and the adjustment formula (7) becomes by substitution of these expressions, and by using the notation of (14),

$$u'_0 = A_0b_0+A_1b_1+A_2b_2+A_3b_3+\&c.\quad(18)$$

If the series to be adjusted is of the zero order, so that all its terms are equal, then in (17) A_1, A_2, &c., will be zero, and in order that the adjusted term u'_0 shall be the same as the corresponding given term $u_0 = A_0$, it is necessary and sufficient that in (18) we shall have $b_0 = 1$. If the series is of the first order, or an arithmetical progression, then in (17) A_2, A_3 &c., will be zero, and since A_0 and A_1 may have any values, it is necessary and sufficient for making $u'_0 = u_0$ that in (18) we should have $b_0 = 1$ and $b_1 = 0$. If the series is of the second order, A_3, A_4 &c., will be zero, and to make $u'_0 = u_0$ we must have $b_0 = 1$, $b_1 = 0$, $b_2 = 0$. So in general, if the given series is of the nth order, the conditions necessary and sufficient to make the adjusted series identical with the given one are

$$b_0 = 1, \qquad b_1 = 0, \qquad b_2 = 0, \; \ldots\ldots \; b_n = 0. \tag{19}$$

This is Schiaparelli's theorem differently proved, and extended to include the case of unsymmetrical coefficients. Since after any number k of successive adjustments the final adjusted series, in the cases supposed, will be the same as the original one, we see that if the conditions (19) are satisfied by the coefficients λ in the original formula (7), they will also be satisfied by the l in the resultant formula (8), and consequently also by the y in the limiting curve (16). The numerical coefficients $1^n, 2^n, 3^n$ &c., of the λ in (14) will become the nth powers of the number i which denotes the rank of the l in (12) and of the y in (16). Thus the conditions $b_0 = 1$ and $b_n = 0$ for the adjustment formula (7), imply for (8) and (12) the conditions

$$\Sigma l = 1, \qquad \Sigma i^n l = 0. \tag{20}$$

Every adjustment formula must obviously be such that if we apply it to a series of given terms which are all equal, the adjusted terms will be equal to the given ones. This requires that the sum of all the λ's in (7) shall be unity, so that we always have $b_0 = 1$. As for b_1, we can reduce it to zero in (16) by changing the place of the origin of coordinates on the axis of X. The coefficients λ in (7) may be regarded as a system of parallel forces acting perpendicularly at intervals of unity along the axis, which is itself regarded as a mathematical lever with fulcrum at λ_0. Thus any term $n\lambda_n$, in the expression for b_1 in (14) is the moment of λ_n about the fulcrum, and b_1 is the moment of the whole system. If the fulcrum coincides with the centre of parallel forces, there is equilibrium, and $b_1 = 0$. In this case, putting $b_0 = 1$ and $b_1 = 0$ in (16), and neglecting the d^2y, d^3y &c., in the denominator, in comparison with y, we get

$$-b^2\left(\frac{dy}{y}\right) + \tfrac{1}{2}\left(b_2 + b_3\right)\left(\frac{d^2y}{y}\right) - \frac{1}{2.3}\left(2b_2 + 3b_3 + b_4\right)\left(\frac{d^3y}{y}\right) + \&\text{c.}$$
$$= \frac{x}{(k+1)dx}. \tag{21}$$

Taking the problem of adjustment in its most general form, supposing the λ to have any arbitrarily assigned values consistent with $b_0 = 1$, the centre of parallel forces will of course be in general not at λ_0, but at some other place. The values of the coefficients in a resultant formula after k applications, however, are independent of the position of the origin or zero point. For example, let the adjustment formula include six terms, whose coefficients are $-\frac{1}{7}, \frac{2}{7}, \frac{4}{7}, \frac{3}{7}, \frac{1}{7}, -\frac{2}{7}$, and let us put λ_0 equal first to the $\frac{2}{7}$, and secondly to the $\frac{3}{7}$, or two intervals to the right of its first place. We thus have the two formulas

$$u'_0 = \tfrac{1}{7}(-u_{-1} + 2u_0 + 4u_1 + 3u_2 + u_3 - 2u_4),$$
$$u'_0 = \tfrac{1}{7}(-u_{-3} + 2u_{-2} + 4u_{-1} + 3u_0 + u_1 - 2u_2),$$

whose coefficients are the same and in the same order, but differently situated with reference to the origin. If the given series is adjusted twice by each of these, the coefficients of the resultant formulas will in both cases be those of the powers of z in the expansion of

$$\left(\frac{-1 + 2z + 4z^2 + 3z^3 + z^4 - 2z^5}{7}\right)^2,$$

so that the two resultant formulas are

$$u''_0 = \tfrac{1}{49}(u_{-2} - 4u_{-1} - 4u_0 + 10u_1 + 26u_2 + 32u_3 + 9u_4 - 10u_5 - 11u_6 - 4u_7 + 4u_8),$$
$$u''_0 = \tfrac{1}{49}(u_{-6} - 4u_{-5} - 4u_{-4} + 10u_{-3} + 26u_{-2} + 32u_{-1} + 9u_0 - 10u_1 - 11u_2 - 4u_3 + 4u_4).$$

The coefficients are the same in both, but the origins or places of u_0 are separated by four intervals, being twice as far as in the original formulas. If each formula were applied k times in succession, the origins in the final resultants would be k times as far apart as they were at first, but both sets of resultant coefficients would still be alike. If therefore we suppose the origin or fulcrum in (7) to be transferred to the center of parallel forces, that is, shifted through an interval b_1 to the right of its first position, the limiting curve will be unchanged, and will still be characterized by the differential equation (16), only its origin will be transferred through a distance $b_1 k dx$ to the right, and the constants b_1, b_2, b_3 &c., of (14) and the i in (20) must now be reckoned from the new origin or center of parallel forces, and we have $b_1 = 0$. Thus equation (16) has lost nothing of its generality by being put in the form (21). The principle of continuity implies that a shifting of the origin of the resultant formula through an interval of $b_1 k$ units will correspond analytically to a shifting of the origin in the original formula through b_1 units, even though b_1 should not be a whole number. In such a case, the numbers i become fractional.

Now if b_2 estimated from the centre of parallel forces, is not zero, neglecting in (21) the differentials of a higher order in comparison with those of a lower one, we have this differential equation of the first order,

$$-b_2\left(\frac{dy}{y}\right)=\frac{xdx}{(k+1)(dx)^2},\qquad \therefore\ \frac{dy}{dx}=\frac{-xy}{b_2(k+1)(dx)^2}.\qquad (22)$$

The denominator $b_2(k+1)(dx)^2$ is a finite constant, and k is an infinity of the second order. (ANALYST, May, 1879 pp. 67 and 72.)

The constant of integration will be determined by the condition $b_0=1$, which, as (20) shows, is in the limiting curve equivalent to $\Sigma y=1$, that is,

$$\frac{1}{dx}\int_{-\infty}^{\infty} ydx=1.\qquad (23)$$

The condition $b_1=0$ is not used for determining a constant. The value of b_1 cannot affect the dimensions or form of the curve, but only its place, or the place of λ_0, along the axis of X.

Next, suppose that besides $b_0=1$ we have $b_2=0$; then (21) reduces to

$$\tfrac{1}{2}b_3\left(\frac{d^2y}{y}\right)=\frac{xdx}{(k+1)(dx)^2},\qquad \therefore\ \frac{d^2y}{dx^2}=\frac{2xy}{b_3(k+1)(dx)^3}.\qquad (24)$$

Here k must be an infinity of the third order, if x is finite and if d^2y is an infinitesimal of the second order as compared with y. There will be two constants of integration, determined by the two conditions, $b_0=1$ and $b_2=0$, which (20) shows to be equivalent at the limit to $\Sigma y=1$ and $\Sigma i^2y=0$, that is to say, since $i=x\div dx$,

$$\frac{1}{dx}\int_{-\infty}^{\infty} ydx=1,\qquad \int_{-\infty}^{\infty} x^2ydx=0.\qquad (25)$$

Likewise when $b_0=0$, $b_2=0$, $b_3=0$, (21) becomes

$$-\frac{1}{2.3}b_4\left(\frac{d^3y}{y}\right)=\frac{xdx}{(k+1)(dx)^2},\qquad \therefore\ \frac{d^3y}{dx^3}=\frac{-2.3xy}{b_4(k+1)(dx)^4}\qquad (26)$$

with the three conditions,

$$\frac{1}{dx}\int_{-\infty}^{\infty} ydx=1,\qquad \int_{-\infty}^{\infty} x^2ydx=0,\qquad \int_{-\infty}^{\infty} x^3ydx=0,\qquad (27)$$

and so on when also $b_4=0$, $b_5=0$ &c. Denoting the coefficients of xy in (22), (24), (26) &c., by a, we have this general proposition; that the limiting curve is always characterized by a linear differential equation of the form

$$a=\frac{1.2.3\ \ldots\ldots\ n(-1)^n}{b_{n+1}(k+1)(dx)^{n+1}},\qquad \frac{d^ny}{dx^n}=axy,\qquad (28)$$

where k is an infinity of the $(n+1)$th order, and the n constants of integration are to be determined by the n conditions

$$\left.\begin{array}{c}\dfrac{1}{dx}\displaystyle\int_{-\infty}^{\infty} ydx=1,\\ \displaystyle\int_{-\infty}^{\infty} x^2ydx=\int_{-\infty}^{\infty} x^3ydx=\ldots\ldots=\int_{-\infty}^{\infty} x^nydx=0.\end{array}\right\}\qquad (29)$$

The origin of coordinates is at the centre of parallel forces for the given coefficients λ, and the constants b are reckoned from that point. The above proposition holds good for the ultimate resultant coefficients of any adjustment formula, and also for the coefficients in the expansion of any given polynomial to the power $k = \infty$, when the sum of the given coefficients is unity. It has long been known that when p and q are positive fractions whose sum is unity, the terms in the expansion of $(p+q)^m$ take the form of the probability curve, answering to the case $n = 1$ in (28). These terms are the same as the coefficients of the powers of z in the expansion of the binomial $(p+qz)^m$. But I am not aware that any writer has hitherto shown that the limiting form of the coefficients in the expansion of a polynomial is also, in general, the probability curve, while in special cases it may be one of the higher curves included under the form (28) and (29).

The order n of the differential equation will always be less than the number of terms which compose the given polynomial, for the conditions $b_0 = 1$, $b_1 = 0$, $b_2 = 0$ &c., are $n + 1$ in number, and to satisfy them all, we must have at least as many as $n + 1$ disposable coefficients λ. It may happen that there will be no limiting curve at all, the consecutive resultant coefficients diverging from each other as k increases, so that the finite differences in (15) do not become differentials. Such cases appear to be distinguished by the constants of integration taking imaginary values. (ANALYST, Sept., 1878, p. 140.)

We have said in connection with (28), that k is an infinity of the $(n+1)$th order. The whole number of terms in a resultant formula is $2km + 1$, as is shown in (8). Since dx is the interval between consecutive terms, and is contained ∞ times in a finite distance on the axis of X, the whole series of resultant coefficients must apparently be regarded as reaching, at the limit, to a distance comparable to the nth power of infinity, which would be the full extent of the limits of integration in (29). Can we say that the whole axis of X may be regarded as reaching not only to the distance ∞, but to ∞^n? If so, it would seem that the higher the order n of the curve, the more remote will the limit be, and that for a given finite value of k, the limiting form will be approached more nearly if n is small than if it is large.

(*To be continued in No.* 6.)